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# Irreducible components of the equivariant punctual Hilbert schemes

Laurent Evain

*Department de Mathematiques, Universite d'Angers, Boulevard Vavoisier, 49045 Angers Cedex 01, France*

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## Abstract

Let  $\mathbb{H}_{ab}$  be the equivariant Hilbert scheme parameterizing the zero-dimensional subschemes of the affine plane invariant under the natural action of the one-dimensional torus  $T_{ab} := \{(t^{-b}, t^a) | t \in k^*\}$ . We compute the irreducible components of  $\mathbb{H}_{ab}$ : they are in one-to-one correspondence with the set of possible Hilbert functions.

As a by-product of the proof, we give new proofs of results by Ellingsrud and Strømme, namely the main lemma of the computation of the Betti numbers of the Hilbert scheme  $\mathbb{H}^l$  parametrizing the zero-dimensional subschemes of the affine plane of length  $l$  (Invent. Math. 87 (1988) 343), and a description of Bialynicki-Birula cells on  $\mathbb{H}^l$  by means of explicit flat families (Invent. Math. 91 (1988) 365). In particular, we give precise conditions when this last description applies.

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## 0. Introduction

The Hilbert scheme  $\mathbb{H}$  parametrizing the zero-dimensional subschemes of the plane  $\text{Spec } k[x, y]$  is a disjoint union of its components  $\mathbb{H}^l$  parametrizing the subschemes of length  $l$ . These Hilbert schemes admit an action of the two-dimensional torus  $k^* \times k^*$  induced by the linear action  $(t_1, t_2) \cdot x^\alpha y^\beta = (t_1 \cdot x)^\alpha (t_2 \cdot y)^\beta$  of  $k^* \times k^*$  on  $k[x, y]$ . If  $T_{ab} = \{(t^{-b}, t^a), t \in k^*\}$  is a one-dimensional sub-torus, the

*E-mail address:* laurent.evain@univ-angers.fr.

closed subscheme  $\mathbb{H}_{ab} \subset \mathbb{H}$  (resp.  $\mathbb{H}_{ab}^l \subset \mathbb{H}^l$ ) parametrizes by definition the zero-dimensional subschemes invariant under the action of  $T_{ab}$ .

In case  $a = 1$ ,  $b = -1$ ,  $\mathbb{H}_{1-1}$  parametrizes the subschemes with homogeneous ideal while in general,  $\mathbb{H}_{ab}$  parametrizes quasi-homogeneous ideals.

The homogeneous case has been extensively studied by Iarrobino [11]. In particular,  $\mathbb{H}_{1-1}$  is smooth and its irreducible components, denoted by  $G_T$ , have been computed.

Though Iarrobino has focused on the homogeneous case, the following two examples illustrate that it may be very useful to study the other  $H_{ab}$ 's.

For any  $l$ , if  $(a, b)$  is chosen general enough,  $\mathbb{H}_{ab}^l$  is a union of points  $p_i$ . One can extract information from this trivial  $\mathbb{H}_{ab}^l$  via its embedding in  $\mathbb{H}^l$ : Ellingsrud and Strømme [4] have computed the Chow group of  $\mathbb{H}^l$  by describing the tangent spaces  $T_{\mathbb{H}, p_i}$  at the various points  $p_i$  and the action of  $k^* \times k^*$  on them.

Another example of the usefulness of a good description of  $\mathbb{H}_{ab}$  when  $(a, b) \neq (1, -1)$  is the work of Brion [3]: the equivariant cohomology of  $\mathbb{H}^l$  can be explicitly described in terms of the equivariant cohomology of *all* the  $\mathbb{H}_{ab}^l$ .

These examples illustrate that information on  $\mathbb{H}_{ab}$  may sometimes be lifted to  $\mathbb{H}$ . In this sense,  $\mathbb{H}_{ab}$  plays a role similar to the Hilbert scheme  $\mathbb{H}_*$  parametrizing the subschemes of the plane supported by a point  $*$  (the papers by Lehn and Nakajima—[13, 14]—are explicit situations where the geometry of  $\mathbb{H}_*$  has proven to be useful).

Summing up, even if one is primarily interested in studying  $\mathbb{H}$ , it is natural to study  $\mathbb{H}$ ,  $\mathbb{H}_*$  and the  $\mathbb{H}_{ab}$ 's as a whole, for their respective geometries are linked.

One of the very basic questions about the schemes  $\mathbb{H}$ ,  $\mathbb{H}_{ab}$  and  $\mathbb{H}_*$  is to determine their irreducible components. The answers are known for  $\mathbb{H}$  (the irreducible components are the smooth subschemes  $\mathbb{H}^l$  by Hartshorne [10] and Fogarty [7]) and for  $\mathbb{H}_*$  (the irreducible components are  $\mathbb{H}_*^l := \mathbb{H}_* \cap \mathbb{H}^l$  by Briançon [2]), but unknown for  $\mathbb{H}_{ab}$ .

As to  $\mathbb{H}_{ab}$ , it cannot be irreducible for there are obviously disconnected subschemes  $\mathbb{H}_{ab}^l := \mathbb{H}_{ab} \cap \mathbb{H}^l$ . These can still be divided into smaller disconnected pieces as follows. The ideal of a subscheme corresponding to a point of  $\mathbb{H}_{ab}^l$  is quasi-homogeneous (homogeneous if  $a = 1, b = -1$ ) and is a direct sum  $I = \bigoplus_{i \in \mathbb{Z}} I_i$  of vector spaces. The codimensions  $d_i$  of the  $I_i$  are fixed on a connected component and verify  $\sum d_i = l$ . Thus  $\mathbb{H}_{ab}^l = \coprod \mathbb{H}_{ab}(H)$ , where  $H: \mathbb{Z} \rightarrow \mathbb{N}$  runs through the functions verifying  $\sum_{i \in \mathbb{Z}} H(i) = l$  and  $\mathbb{H}_{ab}(H)$  parametrises the zero-dimensional subschemes satisfying  $(\dots, d_{-1}, d_0, d_1, d_2, \dots) = H$ . So the natural question concerning the irreducibility of the equivariant Hilbert schemes is: are the natural candidates  $\mathbb{H}_{ab}(H)$  connected and irreducible?

**Theorem 1.** *For any  $H$ , if  $\mathbb{H}_{ab}(H) \neq \emptyset$ , then  $\mathbb{H}_{ab}(H)$  is smooth and connected (hence irreducible).*

This result was already known for  $(a, b)$  general enough (since then  $\mathbb{H}_{ab}^l$  is a union of points) and for the special value  $(a, b) = (1, -1)$  thanks to Iarrobino [11].

After constructing our proof, the work of Haiman and Sturmfels [9] came to our knowledge. The Hilbert schemes  $\mathbb{H}_{ab}(H)$  are multigraded Hilbert schemes and the framework Haiman and Sturmfels constructed give an other perspective on the problems tackled in the present paper.

**On the proof.** The first point is that  $\mathbb{H}_{ab}$  is smooth, as it is the fixed locus of the smooth scheme  $\mathbb{H}$  under the action of the one-dimensional torus  $T_{ab}$  [12]; so we just have to prove the connectedness of  $\mathbb{H}_{ab}(H)$ .

The general plan is as follows. We explain that  $\mathbb{H}_{ab}(H)$  is naturally stratified by locally closed subschemes  $C(E)$ . The strata are parametrized by staircases (or Young diagrams)  $E$  and in each stratum  $C(E)$  there is a particular subscheme  $Z(E)$ . Each stratum is connected as an affine space, so the problem is to link the strata together. This is done in two steps. Firstly, we partially order the strata and we show that each stratum is connected to a smaller stratum except for some special strata characterized by their staircase  $E$ . We then show that there is in fact a unique special stratum. It follows that all strata can be connected to this special stratum and the theorem follows.

The main point is to find a workable condition to connect a stratum  $C(E)$  to a smaller stratum. To this purpose, we describe the tangent space  $T_{Z(E)}^{ab}$  to  $\mathbb{H}_{ab}(H)$  at the special point  $Z(E)$  of the stratum. It is a vector space whose a base is a combinatorial datum associated with  $E$  (Theorem 3). This vector space splits into a direct sum of a positive tangent space  $T_{Z(E)}^{ab+}$  and a negative tangent space  $T_{Z(E)}^{ab-}$ . We describe a sort of “exponential” map  $T_{Z(E)}^{ab+} \rightarrow \mathbb{H}_{ab}$ , which turns out to be an isomorphism on  $C(E)$ . Using this map, a point  $p$  of  $T_{Z(E)}^{ab+}$  corresponds to a subscheme  $X(p)$  of  $\mathbb{A}^2$  whose equations are completely described. We check that for a suitable action of  $k^*$ , if  $p \neq 0$ , then  $\lim_{t \rightarrow 0} tX(p)$  is a subscheme in a smaller stratum (Proposition 18). This allows us to connect a stratum  $C(E)$  to a smaller stratum provided the positive tangent space in  $Z(E)$  is nontrivial.

To conclude, we exhibit an explicit staircase  $E_m$  characterized combinatorially by a property of minimality (Theorem 19). We show that going down from a stratum to a smaller stratum as explained above, the process always ends in  $C(E_m)$ . The existence of a minimal  $E_m$  is a purely combinatorial statement. Its proof, however, requires the characterization of minimality using the positive tangent space. The method here is very much in the spirit of toric varieties [8] where combinatorial statements are proved via algebro-geometric arguments.

**By-products of the proof.** The methods involved in the proof to study  $\mathbb{H}_{ab}$  remain useful with few changes to study  $\mathbb{H}$ . This allows us to recover results by Ellingsrud–Strømme [4,5].

More precisely, the tangent space  $T_{Z(E)}$  to  $\mathbb{H}$  at  $Z(E)$  is a direct sum of the various invariant tangent spaces  $T_{Z(E)}^{ab}$ . Following the argument of [4], the control of  $T_{Z(E)}$  is the key to compute the Betti numbers of  $\mathbb{H}^l$  and we get by our description of  $T_{Z(E)}$  a new proof of their main technical lemma.

This tangent space  $T_{Z(E)}$  splits into a direct sum  $T_{Z(E)}^+ \oplus T_{Z(E)}^-$ , just like the invariant tangent space  $T_{Z(E)}^{ab}$ . And we can still define an embedding  $e : T_{Z(E)}^+ \rightarrow \mathbb{H}$  using the same methods as for the invariant case, that is by exhibiting an explicit flat family on  $T_{Z(E)}^+$ . The image of  $e$  is a Bialynicki-Birula cell  $C(E)$  with respect to a suitable action of the torus  $k^*$  on  $\mathbb{H}$  (Theorem 13). In other words, we have explicit charts for the Bialynicki-Birula cells  $C(E)$  which come from a suitable action of  $k^*$  on  $\mathbb{H}$ . Though it has been obtained by quite different methods, the flat family which defines  $e$  is up to an identification the same family as in [5]. We note that the families involved in [5] are more general than ours: the cells  $C(E)$  depend on the action of  $k^*$  and our method works only for suitable actions whereas Ellingsrud and Strømme describe the cells corresponding to any action. However, the extra cases are false in general: the universal family constructed by Ellingsrud and Strømme to describe  $C(E)$  is not always of constant length. An explicit example is given (Example 16). Consequently, Theorem 13 can be seen as a prolongation of the analysis of Ellingsrud and Strømme: it provides conditions on the action of  $k^*$  under which the description of the cells given in [5] are valid (Section 3.2).

## 1. The stratified subscheme $\mathbb{H}_{ab}(H)$

### 1.1. Disconnected subschemes of $\mathbb{H}_{ab}$

In this section, we introduce the closed subschemes  $\mathbb{H}_{ab}(H) \subset \mathbb{H}_{ab}$ . As claimed in the introduction, for  $H_1 \neq H_2$ ,  $\mathbb{H}_{ab}(H_1) \cap \mathbb{H}_{ab}(H_2) = \emptyset$ .

Recall that  $k$  is an algebraically closed field, that  $a$  and  $b$  are two relatively prime integers, and that  $T_{ab} = \{(t^{-b}, t^a), t \in k^*\}$  is a one-dimensional sub-torus of  $k^* \times k^* =: T$ . Since  $(a, b) \neq (0, 0)$  we can suppose by symmetry  $b \neq 0$ , and since  $(a, b)$  is defined up to sign, we suppose from now on  $b < 0$ . The Hilbert scheme  $\mathbb{H}^l$  parametrizing the zero-dimensional subschemes of  $\text{Spec } k[x, y]$  of length  $l$  is connected and by definition  $\mathbb{H} = \coprod \mathbb{H}^l$ , where  $\coprod$  stands for a disjoint union. There are actions  $T \times \mathbb{H} \rightarrow \mathbb{H}$  and  $T_{ab} \times \mathbb{H} \rightarrow \mathbb{H}$  of the tori on  $\mathbb{H}$ , which in terms of coordinates are given by  $(t_1, t_2)(x^\alpha y^\beta) = (t_1 x)^\alpha (t_2 y)^\beta$ . We denote by  $\mathbb{H}_{ab} \subset \mathbb{H}$  the subscheme parametrizing the subschemes of the plane invariant under the action of  $T_{ab}$ . Then  $\mathbb{H}_{ab} = \coprod \mathbb{H}_{ab}^l$ , where  $\mathbb{H}_{ab}^l := \mathbb{H}_{ab} \cap \mathbb{H}^l$ . One can still separate  $\mathbb{H}_{ab}^l$  into disjoint subschemes by fixing a Hilbert function.

To do this, let us characterize the subschemes of  $\text{Spec } k[x, y]$  which lie in  $\mathbb{H}_{ab}$  by their ideals. Define the degree  $d$  of a monomial by the formula  $d(x^\alpha y^\beta) = -b\alpha + a\beta$ . If  $I$  is an ideal of  $k[x, y]$ , we let  $I_n := I \cap k[x, y]_n$ , where  $k[x, y]_n$  denotes the vector space generated by the monomials  $m$  of degree  $d(m) = n$ . A subscheme  $Z$  is in  $\mathbb{H}_{ab}$ , if and only if its ideal is quasi-homogeneous with respect to  $d$ , i.e.  $I(Z) = \bigoplus_{n \in \mathbb{Z}} I(Z)_n$ .

By semi-continuity, if a subscheme  $Z' \in \mathbb{H}_{ab}$  is a specialization of  $Z \in \mathbb{H}_{ab}$ , then the codimensions of  $I_n(Z)$  and  $I_n(Z')$  in  $k[x, y]_n$  verify  $\text{codim } I_n(Z) \geq \text{codim } I_n(Z')$ . But  $Z$

and  $Z'$ , being in the same connected component of  $\mathbb{H}$ , have the same length  $l = \sum_n \text{codim } I_n(Z) = \sum_n \text{codim } I_n(Z')$ . It follows that the sequence  $H(Z) = (\dots, h_{-1}, h_0, h_1, \dots)$ , where  $h_i = \text{codim } I_n(Z)$ , is constant on the connected components of  $\mathbb{H}_{ab}$  and that  $h_n = 0$  for  $n \gg 0$  and  $n \ll 0$ . If  $H = (\dots, 0, 0, h_r, \dots, h_s, 0, 0, \dots)$  is any sequence, we denote by  $\mathbb{H}_{ab}(H)$  the closed subscheme of  $\mathbb{H}_{ab}$  (possibly empty) parametrizing the subschemes  $Z$  verifying  $H(Z) = H$ . By the above, we have  $\mathbb{H}_{ab} = \coprod \mathbb{H}_{ab}(H)$ .

### 1.2. The case $ab \geq 0$ via a theorem of Bialynicki-Birula

In this section, we introduce the stratification on  $\mathbb{H}_{ab}(H)$ , the special points  $Z(E)$  of the cells  $C(E)$  and we prove the theorem under the condition  $ab \geq 0$ . The strata can be defined in terms of Gröbner bases, using Grassmannians or using a Bialynicki-Birula theorem, as explained in [6]. We recall here the first and the last approach. The theorem follows easily if  $ab \geq 0$ : the stratification on  $\mathbb{H}_{ab}(H)$  is reduced to one stratum, which is an affine space by Bialynicki-Birula [1].

Each cell is associated with a staircase  $E$ : to a staircase  $E$  corresponds a subscheme  $Z(E)$  and the cell  $C(E)$  is the unique cell containing the subscheme  $Z(E)$ . To be more precise, we recall that a staircase is a subset of  $\mathbb{N}^2$  whose complement is stable by addition of  $\mathbb{N}^2$ . In this paper, we will identify freely the monomial  $x^p y^q$  with the couple  $(p, q)$  and therefore the expression “staircase of monomials” will make sense. More generally, we will transpose unscrupulously the definitions between couples of integers and monomials. If  $E$  is a staircase, then the vector space  $I^E$  generated by the monomials which are not in  $E$  is an ideal and conversely, every monomial ideal is an ideal  $I^E$  for a unique staircase  $E$ . The subscheme  $Z(E)$  whose ideal is  $I^E$  is in  $\mathbb{H}_{ab}(H)$  if and only if  $E$  has  $H(i)$  elements  $m$  of degree  $d(m) = i$ .

**The Gröbner bases point of view.** We choose to order the monomials of  $k[x, y]$  by the rule:  $m_1 < m_2$  if  $(d(m_1), d_y(m_1)) < (d(m_2), d_y(m_2))$  for the lexicographic order, where  $d_y$  is defined by  $d_y(x^\alpha y^\beta) = \beta$ . Let  $m_1, m_2, \dots$  be the monomials which do not belong to  $E$ . Fixing once and for all  $a, b$  and  $H = (\dots, 0, 0, h_r, \dots, h_s, 0, \dots)$ , an ideal of  $\mathbb{H}_{ab}(H)$  is in  $C(E)$  if, regarding it as a  $k$ -vector space, it admits a basis  $f_1, f_2, \dots$ , where  $f_i = m_i + R_i$ ,  $R_i$  being a linear combination of monomials strictly smaller than  $m_i$ . The locus  $C(E)$  in  $\mathbb{H}_{ab}(H)$  is non empty exactly when  $E$  has  $h_i$  elements in degree  $i$ .

If  $ab < 0$ , the above order is a monomial order in the sense of Gröbner bases. In this case, the theory of Gröbner bases associates with every ideal in  $k[x, y]$  a monomial ideal called initial ideal and  $C(E)$  is the locus in  $\mathbb{H}_{ab}(H)$  parametrizing the ideals whose initial ideal is  $I^E$ .

**The Bialynicki-Birula point of view.** Let  $X$  be a smooth projective variety over  $k$  admitting an action of the torus  $k^*$ . Suppose that the action has a finite number of fixed points  $x_1, \dots, x_n$ . Let  $T_{X, x_i}^+$  be the part of the tangent space to  $X$  at  $x_i$  where the weights of the  $k^*$ -action are positive, and let  $X_i := \{x \in X, \lim_{t \rightarrow 0} (t.x) = x_i\}$ . Then a

theorem of Bialynicki-Birula asserts that the  $X_i$  are a cellular decomposition of  $X$  in affine spaces and satisfy  $T_{X_i, X_i} = T_{X, X_i}^+$ . In the present case, we are not in a projective situation since the underlying space is the affine space. If  $U \subset X$  is a stable open subset of a projective  $X$ , it is still possible to define cells which are affine spaces associated with the fixed points which lie in  $U$ , but these cells do not always cover  $U$ . In our case, fixing two integers  $p$  and  $q$  with  $ap + bq > 0$ , the torus  $k^*$  acts on  $k[x, y]$  by  $t.x = t^p x$  and  $t.y = t^q y$ . This action induces an action of  $k^*$  on  $\mathbb{H}_{ab}(H)$ . The fixed points of  $\mathbb{H}_{ab}(H)$  under  $k^*$  are the monomial subschemes  $Z(E)$ . Applying the Bialynicki-Birula theorem to the action of  $k^*$  on  $\mathbb{H}_{ab}(H)$ , we get a set of cells. We denote by  $C(E)$  the cell associated with the fixed point  $Z(E)$ . The previous description of these cells insure that they cover  $\mathbb{H}_{ab}(H)$ .

**Proof of the theorem in the case  $ab \geq 0$ .** When the product  $ab$  is nonnegative, there is at most one staircase  $E$  compatible with the Hilbert function  $H$  (i.e. such that  $E$  has  $h_i$  elements in degree  $i$ ). It follows that  $\mathbb{H}_{ab}(H)$  is empty or an affine space  $C(E)$ . The theorem is then obvious.

## 2. Description of the tangent space

Let  $E$  be a staircase,  $T_{Z(E)}$  be the tangent space to  $\mathbb{H}$  at the point  $Z(E)$ ,  $T_{Z(E)}^{ab}$  be the tangent space to  $\mathbb{H}_{ab}$  at the point  $Z(E)$ . In this section, we give a description of  $T_{Z(E)}^{ab}$  and  $T_{Z(E)}$ .

We need some combinatorial vocabulary that we introduce now.

**Clefts and cleft couples.** A cleft for  $E$  is a couple  $(u, v)$  such that  $x^u y^v$  is a monomial in  $I^E$  minimal for the divisibility relation among the monomials of  $I^E$  i.e. such that

$$x^{u'} y^{v'} \in I^E \quad \text{and} \quad (u, v) \neq (u', v') \Rightarrow u < u' \quad \text{or} \quad v < v'.$$

A positive (resp. negative) half-direction is a couple of relatively prime integers  $(f, g)$  with  $f > 0$  (resp.  $f < 0$ ) or  $(f = 0$  and  $g < 0$  (resp.  $g > 0$ )). A couple of points  $(M, N)$  in  $\mathbb{N}^2$  has half-direction  $(f, g)$  if the vector  $\overrightarrow{MN}$  is a positive multiple of  $(f, g)$ . We have a notion of direction by identifying two opposite half-directions. A cleft couple (with respect to  $E$ ) with (half)-direction  $h$  is a couple of elements  $(c, m)$  in  $\mathbb{N}^2$  s.t.  $c$  is a cleft,  $m \in E$ , and  $(c, m)$  has (half)-direction  $h$ .

**The orders  $>_+$  and  $>_-$  on cleft couples.** We put orders on the set of clefts and on the set of cleft couples. The identification between  $x^p y^q$  and  $(p, q)$  gives a lexicographic order  $>_+$  on the monomials of  $k[x, y]$ . The identification between  $x^p y^q$  and  $(q, p)$  gives a lexicographic order  $>_-$ . Any order on the monomials induces an order on the clefts (by restriction), on the couples of monomials (lexicographically) and on the

cleft couples (the restriction of the latter). We still denote the induced orders by  $>_+$  and  $>_-$ .

**Some particular sets of cleft couples and their associated vector spaces.** The combinatorial object used to describe  $T_{Z(E)}$  is the significant cleft couple:

**Definition 2.** Let  $(c, m)$  be a cleft couple with positive (resp. negative) half-direction,  $c'$  be the cleft successor of  $c$  for  $>_+$  (resp. for  $>_-$ ) and  $s$  be the smaller common multiple of  $c$  and  $c'$ . Then  $(c, m)$  is said to be a significant couple if  $m_c^s \notin E$ .

For a cleft couple, we will wonder whether it is significant and what is its direction. Formally, we introduce the sets  $C, \overline{C}, C_{ab}, \overline{C}_{ab}$  containing, respectively,

- the significant cleft couples (with respect to  $E$ ),
- the cleft couples,
- the significant cleft couples with direction  $(a, b)$ ,
- the cleft couples with direction  $(a, b)$ .

We have the obvious inclusions

$$\begin{array}{ccc} C & \rightarrow & \overline{C} \\ \uparrow & & \uparrow \\ C_{ab} & \rightarrow & \overline{C}_{ab} \end{array}$$

For each of the above sets, we can form the vector space on this set, i.e. the vector space whose elements are the formal linear combinations of elements of this set. We denote by  $R, \overline{R}, R_{ab}, \overline{R}_{ab}$  the vector spaces corresponding to  $C, \overline{C}, C_{ab}, \overline{C}_{ab}$ . To the inclusion of sets corresponds the inclusion of vector spaces

$$\begin{array}{ccc} R & \rightarrow & \overline{R} \\ \uparrow & & \uparrow \\ R_{ab} & \rightarrow & \overline{R}_{ab} \end{array}$$

We can now formulate the main result of the section, which is a combinatorial description of  $T_{Z(E)}^{ab}$ . Its proof ends with Corollary 6.

**Theorem 3.** *There is an isomorphism  $T_{Z(E)}^{ab} \simeq R_{ab}$ .*

We start by constructing an injective morphism  $\varphi: T_{Z(E)}^{ab} \rightarrow \overline{R}$ . Next, we will identify the image  $\varphi(T_{Z(E)}^{ab})$  with  $R_{ab}$ .

We have the classical description of tangent spaces on Hilbert schemes  $T_{Z(E)} \simeq \text{Hom}_{k[x, y]}(I^E, k[x, y]/I^E)$ . Let us recall in our situation how to produce the infinitesimal deformation of  $k[x, y]/I^E$  starting from an element  $f \in \text{Hom}_{k[x, y]}(I^E, k[x, y]/I^E)$ .

Let  $V_E$  be the vector space generated by the monomials which are in  $E$ ,  $\pi: V_E \rightarrow k[x, y]/I^E$  the isomorphism induced by the restriction of the projection  $k[x, y] \rightarrow k[x, y]/I^E$ , and  $p: k[x, y]/I^E \rightarrow k[x, y]$  the inverse of  $\pi$ . Let  $\bar{f} = p \circ f: I^E \rightarrow k[x, y]$ . The set of elements  $J = \{m + \varepsilon \bar{f}(m), m \in I^E\}$  is an ideal of  $k[x, y][\varepsilon]/(\varepsilon^2)$ . The quotient  $k[x, y][\varepsilon]/J$  is the flat infinitesimal deformation of  $k[x, y]/I^E$  corresponding to  $f$ .

For every cleft couple  $(c, m)$ , define  $\lambda_{c,m}$  by the formula  $\bar{f}(c) = \sum \lambda_{c,m} m$ .

**Proposition 4.** *The linear map*

$$\varphi: T_{Z(E)} \rightarrow \bar{R}$$

$$f \text{ s.t. } \bar{f}(c) = \sum \lambda_{c,m} m \mapsto \sum \lambda_{c,m}(c, m)$$

is injective and its restriction to  $T_{Z(E)}^{ab}$  factorizes:

$$\begin{array}{ccc} T_{Z(E)}^{ab} & \rightarrow & \bar{R} \\ & \searrow & \uparrow \\ & & \bar{R}_{ab} \end{array}$$

**Proof.** The ideal  $I^E$  being generated by the clefts, the morphism  $f \in \text{Hom}_{k[x,y]}(I^E, k[x, y]/I^E)$  is characterized by the images of the clefts. But these images are themselves characterized by  $\varphi(f)$  so  $\varphi$  is injective.

We can characterize  $T_{Z(E)}^{ab} \subset T_{Z(E)}$  as the subspace containing the vectors fixed under the action of  $T_{ab}$ . If  $f$  is in  $T_{Z(E)}^{ab} \subset T_{Z(E)}$ , one then sees from the description of  $f$  as an infinitesimal deformation of  $k[x, y]/I^E$  that  $\lambda_{c,m}$  is different from zero only if  $(c, m)$  has direction  $(a, b)$ . It follows that  $\varphi(T_{Z(E)}^{ab}) \subset \bar{R}_{ab}$ .  $\square$

The next step is to show the equality  $\varphi(T_{Z(E)}^{ab}) \simeq R_{ab}$ . This is done in two steps. Firstly, we construct a graph  $G$  associated to  $E$ , a vector space  $R_G$  from  $G$  and we show the equality  $\varphi(T_{Z(E)}^{ab}) = R_G$ . We then conclude with the isomorphism  $R_G \simeq R_{ab}$ .

We construct a graph  $G$  from the staircase  $E$ . The set of points is the set  $C_{ab}$  of cleft couples with direction  $(a, b)$ . Significant couples are not the end of any arrow. Nonsignificant cleft couples are the end of exactly one arrow. Let  $(c_1, m_1)$  be nonsignificant with positive half direction, and  $c_2$  be the cleft successor of  $c_1$  for  $>_+$ . If  $m_2 := m_1 \cdot \frac{a_2}{c_1} \in k[x, y]$ , draw an arrow from  $(c_2, m_2)$  to  $(c_1, m_1)$ , otherwise draw an arrow from  $(c_1, m_1)$  to itself. Replacing  $>_+$  by  $>_-$ , you get arrows ending at cleft couples with negative half-direction.

We now have a graph  $G$  with a set of points  $\mathcal{P} = \{p_1, \dots, p_n\}$  and arrows  $f_1, \dots, f_p$ . We denote by  $o(f_i)$  the origin of  $f_i$  and  $e(f_i)$  the end of  $f_i$ . Let  $R_{\mathcal{P}}$  be the vector



space whose base is the set of elements of  $\mathcal{P}$  and  $R_G$  be the sub-vector space of  $R_{\mathcal{P}}$  whose elements are the linear combinations  $\sum \lambda_{p_i} p_i$  verifying  $\lambda_{p_i} = \lambda_{p_j}$  if there is an arrow from  $p_i$  to  $p_j \neq p_i$  and  $\lambda_{p_i} = 0$  if there is an arrow from  $p_i$  to itself.

**Proposition 5.** *If  $ab \leq 0$ , then  $\varphi(T_{Z(E)}^{ab}) = R(G)$ . If  $ab > 0$ , then  $\varphi(T_{Z(E)}^{ab})$  is a subspace of  $R(G)$ .*

**Proof.** We have seen that an element  $f$  in  $T_{Z(E)} = \text{Hom}_{k[x,y]}(I^E, k[x,y]/I^E)$  is characterized by the images  $g_i = f(c_i)$  of the clefts  $c_i$ . Reciprocally, if we prescribe an image  $g_i$  for each cleft  $c_i$ , a compatibility relation insures the existence of  $f$  in  $T_{Z(E)}$  sending  $c_i$  to  $g_i$ : if  $c_i <_+ c_j$  are two clefts such that  $c_j$  is the successor of  $c_i$ , and if  $s$  is the smaller common multiple (s.c.m.) of  $c_i$  and  $c_j$ , then their images  $g_i, g_j$  have to verify

$$g_i \cdot \frac{s}{c_i} = g_j \cdot \frac{s}{c_j}.$$

The element  $g_i$  is a linear combination  $\sum \lambda_{i,k} \overline{m_k}$  and we identify it with the element  $\sum \lambda_{i,k}(c_i, m_k)$ . We do similarly for  $g_j$ . The above relation between  $g_i$  and  $g_j$  translates into relations between the coefficients  $\lambda_{i,k}$  and  $\lambda_{j,l}$ . One sees that these relations are the relations given by the arrows of the graph in the case  $ab \leq 0$  so we are done. In the case  $ab > 0$ , there are at least these relations and possibly some others.  $\square$

**Corollary 6.** *If  $ab \leq 0$ , then  $T_{Z(E)}^{ab} \simeq R_{ab}$ . If  $ab > 0$ , then  $T_{Z(E)}^{ab} = R_{ab} = 0$ .*

**Proof.**  $G$  is the union of its connected components. By construction, these connected components are chains of points  $p_1, p_2, \dots, p_n$  with arrows from  $p_i$  to  $p_{i+1}$  and possibly from  $p_1$  to itself. The vector space  $R(G)$  is then obviously isomorphic to  $R(G')$ , where  $G'$  is the graph with no arrow obtained from  $G$  by keeping the points that are not the end of any arrow. The set of points of  $G'$  is just the set  $C_{ab}$  of significant cleft couples with direction  $(a, b)$  so  $R(G) = R(G') = R_{ab}$  and the first claim of the corollary is a consequence of the previous proposition. In the case  $ab > 0$ ,  $G'$  is empty because there are no significant cleft couples. So  $R(G) = 0$  and the second claim follows again from the last proposition.  $\square$

This corollary obviously concludes the proof of Theorem 3. It also clarifies the structure of  $\mathbb{H}_{ab}$  in some cases. We have already explained that  $\mathbb{H}_{ab}$  is a disjoint union of affine spaces if  $ab \geq 0$ . These affine spaces are particularly simple if  $ab > 0$ .

**Corollary 7.** *If  $ab > 0$ , then  $\mathbb{H}_{ab}(H)$  is empty or reduced to a point.*

**Proof.** There is at most one staircase  $E$  compatible with  $H$ . If  $E$  does not exist, then  $\mathbb{H}_{ab}(H) = \emptyset$ . Otherwise,  $\mathbb{H}_{ab}(H) = C(E)$ . The cell  $C(E)$  is an affine space by Bialynicki-Birula and its dimension is zero by the previous Corollary (6).  $\square$

Following the lines of the proof of Theorem 3 and simply forgetting the arguments concerning the action of the torus  $T_{ab}$ , we have a description of the tangent space  $T_{Z(E)}$  instead of a description of the invariant tangent space  $T_{Z(E)}^{ab}$ .

**Theorem 8.** *The tangent space  $T_{Z(E)}$  to  $\mathbb{H}$  at  $Z(E)$  is isomorphic to the vector space  $R$  whose base is the set of significant cleft couples of  $E$ .*

**Remark 9.** This description of the tangent space implies the main lemma (3.2) of [4]

### 2.1. Definition of the positive and negative tangent spaces

Let  $C_{ab+}$  (resp.  $C_{ab-}$ ) be the set of significant cleft couples with direction  $(a, b)$  and positive (resp. negative) half-direction. Let  $R_{ab+}$  (resp.  $R_{ab-}$ ) be the vector space on the set  $C_{ab+}$  (resp. on  $C_{ab-}$ ). To the decomposition  $C_{ab} = C_{ab+} \amalg C_{ab-}$  corresponds the decomposition  $R_{ab+} \oplus R_{ab-} = R_{ab}$ . Considering the isomorphism  $T_{Z(E)}^{ab} = R_{ab} = R_{ab+} \oplus R_{ab-}$ , we will say that  $R_{ab+}$  (resp.  $R_{ab-}$ ) is the positive (resp. negative) tangent space to  $\mathbb{H}_{ab}$  at  $Z(E)$  and we will write it down  $T_{Z(E)}^{ab+}$  (resp.  $T_{Z(E)}^{ab-}$ ). We define  $T_{Z(E)}^+ := \bigoplus T_{Z(E)}^{ab+}$  where the sum runs over all the directions  $(a, b)$ .

## 3. The exponential map

### 3.1. The invariant case

By the previous section, there is an isomorphism  $T_{Z(E)}^{ab+} \simeq \text{Spec } k[X_{c,m}]$  where the variables  $X_{c,m}$  are in bijection with the cleft couples  $(c, m)$  of  $C_{ab+}$ . The goal of this section is to produce a sort of “exponential” map  $e : T_{Z(E)}^{ab+} \rightarrow \mathbb{H}_{ab}$  which induces an isomorphism between  $T_{Z(E)}^{ab+}$  and the image  $C(E)$ . In other words, this section provides an explicit chart for the cell  $C(E)$ . In the remainder of this section, we will make the assumption that the half-direction  $(a, b)$  verifies  $a > 0$  and  $b < 0$ .

The morphism  $T_{Z(E)}^{ab+} \rightarrow \mathbb{H}$  corresponds to a universal ideal over  $\text{Spec } k[X_{c,m}]$  that we describe now. Let  $c_1 <_+ c_2 <_+ \dots <_+ c_n$  be the clefts of  $E$ . We define a set of monomials  $\Delta_i$  for  $1 \leq i \leq n$  by decreasing induction:  $\Delta_n$  is the set of monomials divisible by  $c_n$  and, for  $i \neq n$ ,  $\Delta_i$  is the set of monomials divisible by  $c_i$  but not divisible by  $c_{i+1}$ . We denote by  $E_i$  the set of significant cleft couples  $(c, m)$  verifying  $c = c_i$ . We define a polynomial  $P(c_i) \in k[X_{c,m}][x, y]$  for each cleft  $c_i$  and a polynomial  $Q(c_i, m) \in k[X_{c,m}][x, y]$  for each  $(c_i, m) \in E_i$  by decreasing induction on  $i$ . When  $i = n$ , we put

$$P(c_n) = c_n.$$

The set  $E_n$  being empty, there is no polynomial  $Q(c_n, m)$  to define. For a general  $i$  and a significant cleft couple  $(c_i, m)$ ,  $g$  the s.c.m. of  $c_i$  and  $c_{i+1}$ , let  $c_{i+k}$  ( $k > 0$ ) be the cleft such that  $m.(g/c_i)$  is in  $\Delta_{i+k}$ . We put

$$Q(c_i, m) = P(c_{i+k}) \cdot \frac{m}{c_{i+k}}$$

and

$$P(c_i) = P(c_{i+1}) \cdot \frac{c_i}{c_{i+1}} + \sum_{m \text{ s.t. } (c_i, m) \text{ is significant}} X_{c_i, m} Q(c_i, m).$$

**Proposition 10.** (a) The ideal  $I_+$  generated by the polynomials  $P(c_i)$  defines a flat family  $Z_+$  of constant length over  $\text{Spec } k[X_{c, m}]$ .

(b) The fiber of  $Z_+$  over the origin is  $Z(E)$ .

**Proof.** When  $X_{c_i, m} = 0$ , for all  $(c_i, m)$ , then  $P(c_i) = c_i$  and  $I_+ = I^E$ . This shows point (b).

To verify (a) we use the theory of Gröbner bases. It suffices to find a monomial order on  $\text{Spec } k[x, y]$  such that, with respect to that monomial order, the  $P(c_i)$  are a Gröbner basis of  $I_+$  over each point and verify  $\text{in}(P(c_i)) = c_i$ . Indeed, if it is the case, the monomials which are in  $E$  form a basis of  $k[x, y]/I_+$  over each point, so the length of the fibers of  $k[x, y]/I_+$  equals the cardinality of  $E$ . Each fiber being supported by the origin of  $\mathbb{A}^2$ , the assertion on the length of the fibers implies the flatness. We choose the monomial order  $>_-$ . According to the Buchberger algorithm, to verify that the  $P_i$  form a Gröbner basis, we must verify that if  $c_i$  and  $c_{i+1}$  are two consecutive cleft couples, and if  $g$  is the s.c.m. of  $c_i$  and  $c_{i+1}$ , then the remainder of a division of  $h = \frac{g}{c_i}P(c_i) - \frac{g}{c_{i+1}}P(c_{i+1})$  by the  $P(c_k)$  is zero. But

$$h = \frac{g}{c_i} \sum_{m \text{ s.t. } (c_i, m) \text{ is significant}} X_{c_i, m} Q(c_i, m).$$

By construction,  $\frac{g}{c_i}Q(c_i, m)$  is a product of a polynomial  $P(c_{i+k})$  by a monomial. So the remainder after division is zero.  $\square$

**Theorem 11.** The morphism  $e: T_{Z(E)}^{ab+} \rightarrow \mathbb{H}$  defined by  $I_+$  induces an isomorphism  $T_{Z(E)}^{ab+} \simeq C(E)$ .

**Proof.** According to the Gröbner bases considerations in the last proof, the image of  $e$  verifies  $\text{Im}(e) \subset C(E)$ . If  $e$  is an embedding, then  $\text{Im}(e) = C(E)$  and the theorem is proved. Indeed, suppose that  $e$  is an embedding and that there exists  $p \in C(E) - \text{Im}(e)$ . Then, considering the action of  $T = k^* \times k^*$ , the closure of the orbit  $\overline{T.p}$

meets  $Im(e)$  at  $Z(E)$ . It follows that  $T_{p \cup Im(e)} \subset C(E)$  is singular at  $Z(E)$ . Counting the dimensions of the tangent spaces gives the inequality

$$\dim T_{C(E), Z(E)} \geq \dim T_{T_{p \cup Im(e)}, Z(E)} > \dim Im(e) = \dim T_{Z(E)}^{ab+}.$$

This contradicts the equality of the first and last tangent spaces asserted by Bialynicki-Birula. We will conclude with the construction of a morphism  $i: \mathbb{H}^{card(E)} \rightarrow \mathbb{P}^N$  such that  $i \circ e: T_{Z(E)}^{ab+} \rightarrow \mathbb{P}^N$  is an embedding.

Fix now  $n > 0$  and let  $K_n \subset k[x, y]$  be the vector space generated by the monomials  $m$  of degree  $d(m) \leq n$ . If  $n$  is big enough, and if  $I$  is an ideal in  $\mathbb{H}^{card(E)}$ ,  $I \cap K_n \subset K_n$  is a vector space of codimension  $card(E)$ , corresponding to a point in the Grassmannian  $G = G(\dim K_n - card(E), K_n)$ . By the usual Plucker embedding,  $G$  embeds in the projective space  $\mathbb{P}(\Lambda^{\dim K_n - card(E)} K_n)$ . Summing up, we have constructed a morphism  $i: \mathbb{H}^{card(E)} \rightarrow \mathbb{P}^N = \mathbb{P}(\Lambda^{\dim K_n - card(E)} K_n)$  such that, if  $g_1, \dots, g_p$  is a basis of  $I \cap K_n$ ,  $i(I)$  is the one-dimensional space generated by  $g_1 \wedge g_2 \cdots \wedge g_p$ .

Define the set of monomials  $C_n$  by

$$C_n := \{m = x^\alpha y^\beta \notin E \text{ such that } d(m) \leq n\}.$$

Each  $m$  of  $C_n$  is in one sector  $\Delta_i$ . Put  $f_m = P(c_i) \cdot \frac{m}{c_i}$ . A division of  $f_m$  by the elements  $f_{m'}$  smaller than  $f_m$  with respect to the order  $>_-$  is:

$$f_m = \sum_{d(m')=d(m), m' <_- m} q_{m'} f_{m'} + g_m, \quad \text{where } q_{m'} \in k.$$

The remainder  $g_m$  writes down

$$g_m = m + \sum_{m_i \in E, d(m_i)=d(m)} \mu_{m, m_i} m_i.$$

By the above, if we have chosen  $n$  big enough,  $i \circ e$  is given in Plücker coordinates by the coefficients of  $\bigwedge_{m \in C_n} g_m$ . The coordinate corresponding to the term  $\bigwedge_{m \in C_n} m$  equals 1 and  $i \circ e$  is then a morphism from  $Spec k[X_{c, m}]$  to an affine space  $Spec k[Y_1, \dots, Y_p]$ . If  $(c, m)$  is a significant cleft couple, the coordinate corresponding to the factor  $m \wedge (\bigwedge_{m' \in C_n, m' \neq c m'})$  is  $\mu_{c, m} = X_{c, m} + R_{c, m}$ , where  $R_{c, m}$  is a polynomial in the variables  $X_{c', m'}$ , with  $c' <_- c$  or  $(c' = c \text{ and } m' <_+ m)$ . This means that we can order the variables  $X_{c, m}$  and the variables  $Y_1, \dots, Y_p$  such that the morphism

$$i \circ e: Spec k[X_{c, m}] =: Spec k[X_1, \dots, X_q] \rightarrow Spec k[Y_1, \dots, Y_p]$$

is given by  $Y_1 = X_1$  and for  $2 \leq i \leq q$ ,  $Y_i = X_i + R_i$  where  $R_i$  is a polynomial in the variables  $X_j$ ,  $j < i$ . This shows that  $i \circ e$  is an embedding.  $\square$

### 3.2. The general case

In the previous section, we have defined a map  $e: T_{Z(E)}^{ab+} \rightarrow \mathbb{H}_{ab}$  which induces an isomorphism  $e: T_{Z(E)}^{ab+} \rightarrow C(E)$ . In this section, we extend the results from the invariant case to the general case. Explicitly, we construct a map  $T_{Z(E)}^+ \rightarrow \mathbb{H}$ , which induces an isomorphism between  $T_{Z(E)}^+$  and a Bialynicki-Birula cell  $C(E)$  of  $\mathbb{H}$  and we explain the link between this description of  $C(E)$  and the one of [5]. Note that this section is not useful for the proof of Theorem 1.

Choose  $a < 0$  and  $b < 0$  relatively prime. The one-dimensional torus  $k^* = T_{ab}$  acts on  $\mathbb{H}$  with the monomial subschemes as the only invariant subschemes by (7), so we can apply Bialynicki-Birula to define cells from this action. To be concrete,

$$C(E) = \left\{ \text{subschemes } X \subset \mathbb{A}^2 \text{ such that } \lim_{t \rightarrow 0} (t^{-b}, t^a).X = Z(E) \right\}. \quad (1)$$

Choose a variable  $X_{c,m}$  for each cleft couple  $(c, m)$  with positive half-direction. Define as in the previous section polynomials  $P(c_i) \in k[X_{c,m}][x, y]$  for each cleft  $c_i$ ,  $Q(c_i, m) \in k[X_{c,m}][x, y]$  for each significant  $(c_i, m)$ . Explicitly, with the notations of the previous section

$$P(c_n) = c_n,$$

$$Q(c_i, m) = P(c_{i+k}) \cdot \frac{m}{c_{i+k}}$$

and for  $i < n$ ,

$$P(c_i) = P(c_{i+1}) \cdot \frac{c_i}{c_{i+1}} + \sum_{m \text{ s.t. } (c_i, m) \text{ is significant}} X_{c_i, m} Q(c_i, m).$$

**Proposition 12.** *The ideal  $I_+$  generated by the  $P(c_i)$  defines a flat family  $Z_+$  of constant length over  $\text{Spec } k[X_{c,m}]$ .*

**Proof.** This is exactly the same as for Proposition 10, except for the proof of flatness which is not the same since the family is no longer supported by the origin. To conclude that  $Z_+$  is flat over  $\text{Spec } k[X_{c,m}]$  knowing that the fibers have constant length, we will check that  $Z_+$  is a projective family, i.e. that the composition morphism

$$Z_+ \rightarrow \text{Spec } k[X_{c,m}] \times \mathbb{A}^2 \rightarrow \text{Spec } k[X_{c,m}] \times \mathbb{P}^2$$

is a closed embedding, where the first arrow is the closed embedding defining  $Z_+$  and the second arrow is defined by the immersion of the affine plane in the projective

plane. Let  $G$  be the reduced scheme associated with  $Z_+$ . Since being closed is a topological property, we simply need check that  $G \subset \operatorname{Spec} k[X_{c,m}] \times \mathbb{P}^2$  is closed. We already know that  $G \subset \operatorname{Spec} k[X_{c,m}] \times \mathbb{A}^2$  is closed, therefore it suffices to show that the closure  $\overline{G}$  of  $G$  in  $\operatorname{Spec} k[X_{c,m}] \times \mathbb{P}^2$  does not intersect the subset  $\operatorname{Spec} k[X_{c,m}] \times (\mathbb{P}^2 \setminus \mathbb{A}^2)$ . The ideal of  $G$  in  $\operatorname{Spec} k[X_{c,m}] \times \mathbb{A}^2$  contains an ideal  $(x, y^l + a_{l-1}y^{l-1} + \cdots a_0y^0)$ , where  $a_i$  is a polynomial in the indeterminates  $X_{c,m}$ . The closure  $\overline{G}$  is included in the subscheme defined by the homogeneous ideal  $(x, y^l + a_{l-1}y^{l-1}h + \cdots a_0y^0h^l)$ . Thus  $\overline{G}$  does not meet the relative line  $\operatorname{Spec} k[X_{c,m}] \times (\mathbb{P}^2 \setminus \mathbb{A}^2)$ .  $\square$

**Theorem 13.** *The morphism  $e: T_{Z(E)}^+ = \operatorname{Spec} k[X_{c,m}] \rightarrow \mathbb{H}$  defined by  $I_+$  induces an isomorphism  $T_{Z(E)}^+ \simeq C(E)$ .*

**Proof.** The proof is essentially the same as that of Proposition 11, with a few changes left to the reader.  $\square$

**Remark 14.** When  $a < 0$  and  $b < 0$ , we have parametrized the cells  $C(E)$  defined in Section 3.2, formula (1) using the positive tangent space. If  $a > 0, b > 0$ , we can clearly by the same method parametrize the cells using the negative tangent space.

We now explain the link between this last theorem and Theorem 2 of [5].

To describe a Bialynicki-Birula cell  $C(E)$  of  $\mathbb{H}$ , we have produced a set of variables  $C^+ = \{X_{c,m}\}$ , a universal family  $Z_+$  over  $\operatorname{Spec} k[X_{c,m}]$  inducing an isomorphism  $C(E) \simeq \operatorname{Spec} k[X_{c,m}]$ . On the other hand, Ellingsrud and Strømme construct a set of weighted variables  $\{s_{ij}\}$ , a universal family  $Z$  over  $\operatorname{Spec} k[s_{ij}]$  inducing an isomorphism  $C(E) \simeq \operatorname{Spec} k[s_{ij}]$ . (To give a rough idea of  $Z$ , they considered a nonminimal resolution of the ideal  $I^E$ , and they deformed the matrix of the resolution using the variables  $s_{ij}$ . The ideal of  $Z$  is generated by the maximal minors of this matrix). The following proposition (whose proof is omitted) explains that, up to sign, we can identify the variables  $s_{ij}$  and  $X_{c,m}$  such that  $Z = Z_+$ .

To state the identification, we need to be a little more precise since the Bialynicki-Birula stratification of  $\mathbb{H}$  depends in general on the choice of a one parameter subgroup  $\psi: k^* \rightarrow T$ . Considering for instance all the one-parameter subgroups  $\psi = \psi_{ab}: t \mapsto (t^{-b}, t^a)$ , with  $ab > 0$ , there are two distinct stratifications, one for the subgroups  $\psi_{ab}$  with  $a > 0, b > 0$  described by Theorem 13 and one for the subgroups with  $a < 0, b < 0$  described via Remark 14.

Ellingsrud and Strømme deal with every possible subgroup  $\psi$ . Though the cells depend on  $\psi$ , the number of cells is fixed since the cells are parametrized by staircases as in our case. To parameterize such a cell  $C_\psi(E)$ , they construct a set of variables  $s_{ij}$  and they give to each variable a weight  $n_{ij}$  which depends on  $\psi$ . Denoting by  $\Delta(\psi)$  the set of variables with negative weight, they show the isomorphism  $C_\psi(E) \simeq \operatorname{Spec} k[s_{ij}, s_{ij} \in \Delta(\psi)]$ .

In case  $\psi = \psi_{ab}$ , with  $ab > 0$ , we can establish the link between the two theorems. Using symmetry, we can concentrate on the case  $a < 0$  and  $b < 0$ .

In this particular case, the variables  $s_{ij}$  with negative weight are the variables  $s_{ij}$  with  $(i, j) \in \Delta_1$  with the notations of [5]. We can define a bijection  $\varphi: C^+ \rightarrow \Delta(\psi)$  as follows. If  $X_{c,m} \in C^+$ , let  $l > 0$  be the smallest integer such that  $x^l \cdot m \notin E$ . Using the numeration of the monomials of [5], we have  $cx^l = d_j$  and  $mx^l = d_i$ . We let  $\varphi(X_{c,m}) = s_{ij}$ . We are now ready to identify the Ellingsrud–Strømme family  $Z$  and our  $Z^+$ .

**Proposition 15.** *There exists an application  $\varepsilon: C^+ \rightarrow \{1, -1\}$  such that, if*

$$\eta: \text{Spec } k[s_{ij}, s_{ij} \in \Delta(\psi)] \rightarrow \text{Spec } k[X_{c,m}]$$

*is the morphism defined by  $X_{c,m} \mapsto \varepsilon(X_{c,m})\varphi(X_{c,m})$ , then*

$$Z = Z_+ \times_{\text{Spec } k[X_{c,m}]} \text{Spec } k[s_{ij}, s_{ij} \in \Delta(\psi)].$$

The theorem of Ellingsrud and Strømme is more general than ours, since they describe the cells  $C_\psi(E)$  defined by the action of any one parameter subgroup  $\psi$  of  $k^* \times k^*$  whereas we only consider the action of sub-tori  $\psi_{ab}$  with  $ab > 0$ . However, the following example shows that the fibers of the family  $Z$  they produced is not of constant length in general, and it is not possible to associate a morphism to the Hilbert scheme with  $Z$ .

**Example 16.** Consider the one parameter subgroup  $t \mapsto (t^4, t^5)$ , and, following the procedure of [5], construct a family whose fiber over the origin is the subscheme  $Z(E)$  of length 6, where  $I^E = (y^3, xy, x^4)$ . The fiber of  $Z$  over the point with all coordinates  $s_{ij} = 0$  but  $s_{03} = 1$ ,  $s_{74} = 1$  is of length 7 since a Gröbner basis with respect to the homogeneous order with  $x > y$  is  $xy^2 + y^3, x^2y + xy^2, x^3 + x^2y - xy - y^2, y^4 - y^3$ .

Other examples suggest that Theorem 2 of Ellingsrud and Strømme is correct for a one parameter subgroup  $\psi = \psi_{ab}$  if  $(a < 0 \text{ and } b > 0)$  or  $ab \geq 0$ . In the remaining cases  $a > 0$ ,  $b < 0$ , the fibers of the family  $Z$  they constructed are sometimes of constant length, sometimes not. One can understand which cases are correct as follows. Fix a staircase  $E$  and a torus  $\psi = \psi_{ab}$  with  $a > 0$ ,  $b < 0$ . If there exists a torus  $\psi' = \psi_{a'b'}$  with  $a'b' > 0$  such that  $C_\psi(E)$  is an affine subspace of  $C_{\psi'}(E)$ , then one can prove that the fibers of  $Z$  have constant length using Theorem 13 and Remark 14. In substance, the argument is that the cell is a priori defined with a “bad torus”  $\psi$ , but the same call could be defined with a “good torus”  $\psi'$  and we know that the fibers of  $Z$  have constant length for a “good torus”. For a concrete example, let  $E$  be the staircase such that  $I^E = (x^2, y)(a, b) = (1, -2)$ ,  $(a', b') = (1, 1)$ ,  $(c_0, m_0) = (y, x)$ . Then,  $C_\psi(E)$  is an affine subspace of  $C_{\psi'}(E)$ . Indeed, the exponential isomorphism  $e: \text{Spec } k[X_{c,m}] \rightarrow C_{\psi'}(E)$  that parametrizes  $C_{\psi'}(E)$  restricts to an embedding  $r_e: \text{Spec } k[X_{c_0, m_0}] \rightarrow C_{\psi'}(E)$  and  $C_\psi(E)$  is the image  $\text{Im}(r_e)$  of this restriction.

Following [5], the base of the universal family  $Z$  is  $\text{Spec } k[s_{31}]$  and, up to the identification  $X_{c_0, m_0} = s_{31}$ ,  $Z$  is the universal family which defines  $r_e$ . In particular, its fibers are of constant length. Summing up, there are cases where the correctness of  $Z$  can be proved thanks to another torus  $\psi'$ . However, the existence of such a torus  $\psi'$  requires some combinatorial properties on the couple  $(E, \psi)$ , which are not satisfied for a “general”  $(E, \psi)$ . Several examples have been computed. Whenever it was not possible to use the trick of changing the torus to prove the correctness of  $Z$ , the fibers of  $Z$  were not of constant length (Example 16 is the simplest example for which the trick does not work).

#### 4. Connecting the strata

In this section, we suppose  $a > 0$  and  $b < 0$ . The monomials of  $k[x, y]$  can be ordered in an infinite sequence  $m_0 < m_1 < \dots$  with respect to the order  $>$  defined by  $m < m'$  if  $(d(m), d_y(m)) < (d(m'), d_y(m'))$  for the lexicographic order on  $\mathbb{N}^2$ . For a staircase  $E$ , let  $S_E$  be the function from  $\mathbb{N}$  to  $\mathbb{N}$  defined by  $S_E(k) =$  number of monomials in  $E$  smaller or equal to  $m_k$ .

**Definition 17.** We define a partial relation  $>$  on staircases by  $E > F$  if  $\forall k, S_E(k) > S_F(k)$ .

The goal of this section is to show the following proposition:

**Proposition 18.** Let  $E$  be a staircase such that  $T_{Z(E)}^{ab+} \neq 0$ . Then there exists a staircase  $F$  verifying  $F < E$  and  $\overline{C(E)} \cap C(F) \neq \emptyset$ .

**Proof.** Choose a point  $p \in \text{Spec } k[X_{c,m}]$  different from the origin  $O$ . There exists a monomial  $m$  which is in  $I^E$  but not in  $I_+(p)$  (otherwise  $I^E \subset I_+(p)$ , and even  $I^E = I_+(p)$  since the subschemes defined by  $I^E$  and  $I_+(p)$  have the same length, which is impossible since  $e(O) \neq e(p)$  by 11). Take such an  $m$  minimal. The element  $g_m(p) \in I_+(p)$  defined in the proof of Theorem 11 has its terms in  $E$  except the initial term  $m$ . There is at least one term different from  $m$  since  $m \notin I_+(p)$  by definition. Consider the subscheme  $X(p)$  defined by the ideal  $I_+(p)$ . If the torus  $k^*$  acts on  $\text{Spec } k[x, y]$  by  $t.x = tx, t.y = y$ , then the scheme  $X(\infty) = \lim_{t \rightarrow \infty} t.X(p)$  is not in  $C(E)$  since its ideal contains the monomial  $\lim_{t \rightarrow \infty} t.g_m(p) \in E$ . If  $F$  is the staircase of  $X(\infty)$ , then  $\overline{C(E)} \cap C(F) \neq \emptyset$ . This nonvacuity implies  $E > F$  ([6] or [15]).  $\square$

#### 5. Uniqueness of the final stratum and conclusion of the proof

The connectedness of  $\mathbb{H}_{ab}(H)$  has been proved in the first section under the condition  $ab \leq 0$ . In the remaining cases, one can suppose  $a > 0$  and  $b < 0$  and we have stratified  $\mathbb{H}_{ab}(H)$  by affine spaces  $C(E)$ , ordered the staircases parametrizing the



strata and we have proven that  $\overline{C(E)} \cap C(F) \neq \emptyset$  for some staircase  $F < E$  provided that  $T_{Z(E)}^{ab+} \neq 0$ . The stratum  $C(F)$  itself is connected to a smaller stratum  $C(G)$  if  $T_{Z(F)}^{ab+} \neq 0$ . Carrying on, we can go down from stratum to stratum while the positive tangent space at the special points is nontrivial. The process stops after a finite number of steps since the number of possible staircases is finite. In fact, the stratum  $C(E_m)$  in which the process stops does not depend on the intermediate degenerations. The staircase  $E_m$  parametrizing this final stratum is characterized by the following theorem.

Recall that a staircase  $E$  is compatible with  $H = (\dots, h_{-1}, h_0, h_1, \dots)$  if  $C(E) \cap \mathbb{H}_{ab}(H) \neq \emptyset$  or equivalently if  $E$  has  $h_i$  elements in degree  $i$ .

**Theorem 19.** *Let  $S$  be the set of staircases compatible with  $H$ . If  $S \neq \emptyset$ , then there exists in  $S$  a staircase  $E_m$  such that:  $\forall E \in S, E_m \leq E$ .*

To conclude the proof of the connectedness, it clearly suffices to prove this last theorem and to show that the path through the strata always stops in that special stratum  $C(E_m)$ .

Since a minimal staircase  $E$  verifies  $T_{Z(E)}^{ab+} = 0$  by the previous section, and since we stop when  $T_{Z(E)}^{ab+} = 0$ , we just have to show that there is a unique staircase  $E \in S$  such that  $T_{Z(E)}^{ab+} = 0$ , which we prove by induction on the cardinal of  $E$ .

The case  $\text{card}(E) = 1$  is trivial. A general  $E$  is the “vertical collision” of the bottom row  $L$  and of a residual staircase  $U$  determined by its monomials:  $U := \{m \text{ such that } ym \in E\}$ . It is in fact sufficient to show that the bottom row  $L$  of  $E$  is completely determined. Indeed, suppose that  $L$  is known. The positive tangent space at  $Z(U)$  injects in the positive tangent space at  $Z(E)$  by sending  $\sum \lambda_{c,m}(c, m)$  to  $\sum \lambda_{c,m}(yc, ym)$ , so  $T_{Z(U)}^{ab+} = T_{Z(E)}^{ab+} = 0$ . It follows that  $U$  is completely determined by induction (since its Hilbert function is determined by that of  $E$  and  $L$ ).

The next two lemmas complete the proof. They explain that the bottom row  $L$  of a staircase  $E$  with trivial positive tangent space and compatible with  $H$  is determined: the maximal integer  $k$  such that  $x^k \in E$  is the maximum integer such that  $H(-kb - a) < H(-kb)$ . More precisely, Lemma 20 shows that the bottom row contains all the monomials  $x^k$  such that  $H(-kb - a) < H(-kb)$  and Lemma 20 shows that it cannot contain a bigger monomial.

**Lemma 20.** *Let  $E$  be a staircase compatible with  $H$ . Let  $x^k$  be the maximal power of  $x$  such that  $x^k \in E$  and let  $\delta = d(x^k) = -kb$  be its degree. If  $H(\delta) \leq H(\delta - a)$ , then  $T_{Z(E)}^{ab+} \neq \{0\}$ .*

**Proof.**  $E$  contains a monomial  $m$  of degree  $\delta - a$  such that  $ym \notin E$ . Otherwise,  $\varphi: m \mapsto ym$  would be an injective application from the set  $E(\delta - a) :=$

$\{m \in E, d(m) = \delta - a\}$  to  $E(\delta)$ . The map  $\varphi$  is not surjective since  $x^k \notin \text{Im}(\varphi)$ . Thus,  $\text{card}(E(\delta)) = H(\delta) > H(\delta - a) = \text{card}(E(\delta - a))$ , contradicting the hypothesis.

Now choose  $l \in \mathbb{N}$  maximal such that  $c := \frac{ym}{x^l} \in k[x, y] - E$ . By construction,  $(c, x^{k-l})$  is a cleft couple, which shows  $T_{Z(E)}^{ab+} \neq 0$ .  $\square$

**Lemma 21.** *Let  $\delta = -kb$  be an integer. If  $H(\delta - a) < H(\delta)$ , then for any staircase  $E$  compatible with  $H$ , the element  $x^k$  of degree  $\delta$  is in the bottom row of  $E$ .*

**Proof.** If  $x^k$  were not in  $E$ , then to each element  $x^\alpha y^\beta \in E$  of degree  $\delta$  would be associated the element  $x^\alpha y^{\beta-1} \in E$  of degree  $\delta - a$ . A count of these elements would then show  $H(\delta - a) \geq H(\delta)$ .  $\square$

**Remark 22.** The existence of a minimal staircase with respect to the partial order on staircases in Theorem 19 is a purely combinatorial statement which has required an algebro-geometric proof. I have not found a combinatorial proof simpler than the given argument.

**Remark 23.** One can deduce from the proof a construction of the minimal staircase  $E_m$  of Theorem 19 associated with the Hilbert function  $H$ . Indeed,  $E_m$  is the vertical collision of its bottom row and of the residual staircase  $U$  which is the minimal staircase associated with a function  $H'$ . The length  $k$  of the bottom row is explicitly given as the greatest integer such that  $H(-kb - a) < H(-kb)$  and  $H'$  is defined by

- $H'(i) = H(i + a) - 1$  if  $b$  divides  $i + a$  and  $i + a \leq kb$
- $H'(i) = H(i + a)$  otherwise.

The inductive construction of  $E_m$  follows immediately.

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